{...}

The machine stops?

(i)

There is a story, known to every child — at least, to every mathematical child — about a day in 1917, when G.H. Hardy went down into London to visit his friend the Indian prodigy Ramanujan. As Hardy tells it, he took a cab, and when he got there and greeted Ramanujan, he mentioned that the number of the cab had been 1729, and he was afraid that wasn't very interesting. "No, Hardy," Ramanujan instantly replied, "it is a *very* interesting number. It is the smallest number that is the sum of two cubes in two different ways."[1]

Many years after this incident, I found myself sitting at a bar one morning around 2 a.m., talking to an attractive young lady. We were talking, mainly, about cleaning restaurants, since (of course) that was why we were there: the place had closed, the coke-dealing bartenders had gone home, the party was over, and now we were responsible for the mess they'd left behind them; after we finished our coffee, at least. I made some ironic comment about how my mathematical education had prepared me for this task, and she remarked her complete ignorance in that regard. "I don't even know my times tables," she confessed. "Ah," I said, "you mean the ones with multiplication in them." "Yeah," she said. "I don't know anything. For instance that number there." She pointed at a white plastic five gallon barrel on the floor behind the bar. It seemed to have originally been intended to hold pickles, but now it was employed as a trash can.

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[1] Once stated this fact is usually obvious to any mathematical child: 1729 = 1728 +1 = 1000 + 729. — Hardy later devoted a section of his classic treatise *The Theory of Numbers* to the more general problem of numbers which can be expressed as two cubes in n different ways.

"That number probably has a story behind it." I looked at the barrel, and saw that there was some kind of serial number printed at the top. The number was 9232, and for the first, last, and only time in my life, I felt just like Ramanujan. "Yes," I said. "It certainly does."

(ii)

Pick a number (as they say), any number — meaning, a natural number, a positive integer. If it is even, divide by two. If it is odd, multiply by three and add one. Repeat, and if you reach one, stop. Does this process always terminate?

Observe first that

$$1 \rightarrow 4 \rightarrow 1$$

and therefore that it is more natural to ask whether the process terminates in a loop; this is confirmed by extending the domain to negative integers, where the list

$$-1 \rightarrow -1$$

$$-5 \rightarrow -7 \rightarrow -5$$

$$-17 \rightarrow -25 \rightarrow -37 \rightarrow -55 \rightarrow -41 \rightarrow -61 \rightarrow -91 \rightarrow -17$$

appears to be exhaustive.[2] (I employ the convention of skipping the even numbers introduced and divided out in the intermediate steps.)

We then observe that the first few positive odd numbers reduce rapidly:

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[2] It is absolutely typical of the investigation that I had an elegant proof of this proposition which, at the very last moment, sprung a leak I've never managed to plug.

3 —> 5 —> 1

7 —> 11 —> 17 —> 13 —> 5 —> 1

15 —> 23 —> 35 —> 53 —> 5 —> 1

19 —> 29 —> 11 —> ...

21 —> 1

23 —> 35 —> ...

25 —> 19 —> ...

and that once we've arrived at a number we've already encountered, we can stop.

However (as I ascertained a couple of minutes after the problem was first explained to me, working through examples in my head)

27 —> 41 —> 31 —> 47 —> 71 —> 107 —> ...

and in fact though this process does terminate at 1, it takes 41 steps (counting only the odd numbers.)

The largest odd number in this trail is 3077, and thus the largest even number is 9232.

Say that m arrows n (whether even or odd) if the process carries m into n; thus 3 arrows 5, 27 arrows 214 arrows 107, etc. Of the first million positive integers, then, 394059 arrow 9232. Of the first 10 million, 3935382. Of the first hundred million, 39311437. It is the Kevin Bacon of this graph. Asymptotically a bit more than 39 percent of all positive integers arrow 9232.

So, yes. This number does have a story behind it.

(iii)

Why should an arbitrary positive integer arrow 1?

Let us divide this into two separate propositions. We note first that if we commence the process with some arbitrary positive integer, either it increases indefinitely or it reaches an upper bound. In the latter case, obviously, the trail must terminate in a loop. So we want to prove that (1) every integer arrows some loop, and (2) that the only positive loop is

$$1 \longrightarrow 1$$

This suggests two natural generalizations of the problem: first, to negative integers, as already indicated; second, to rational numbers, since for any quotient of integers with numerator not divisible by 2 and denominator not divisible by 2 or 3 we can iterate the same mapping, e.g.

$$3/5 \longrightarrow 7/5 \longrightarrow 13/5 \longrightarrow 11/5 \longrightarrow 19/5 \longrightarrow 31/5 \longrightarrow 49/5 \longrightarrow 19/5$$

(For rational p/q this is equivalent to asking the question for integers with the mapping $p \longrightarrow (3p + q), p/2$.)

In the case of the negative integers, there appear to be a finite number of possible loops (enumerated above); in the case of the rationals, this isn't so.

Another natural generalization is to different multipliers, e.g. substituting 5 for 3, but it is easy to see that most of these variations blow up: starting with 7, for example, we have after a thousand iterations

177803902857476202974387921101853813205036988404708559
908019638482321417529129636281709 33

and give up.

A halfassed probability argument shows when the process should be bounded: if we pick an odd number n at random, when we multiply by 3 and add 1 we get an even number every time, a number divisible by 4 half the time,[3] a number divisible by 8 one quarter of the time, etc., so the expected value of the transform is

$$n \to \frac{1}{2} \cdot \frac{3n+1}{2} + \frac{1}{4} \cdot \frac{3n+1}{4} + \frac{1}{8} \cdot \frac{3n+1}{8} + \ldots$$

$$= (3n+1)\left(\frac{1}{4} + \frac{1}{16} + \ldots\right) = n + \frac{1}{3}$$

meaning n stays about the same size on average. With a multiplier greater than 3, however, this will not be the case.

As one would expect of a conjecture such as this, dwarves toiling in the Mines of Moria have verified it for every positive number less than some prodigious figure,[4] with the usual result that initial uncertainty about the truth or falsehood of the assertion has long since been replaced by frustration at being unable to see how to prove it. — I have personally tested random integers up to bounds on the order of 2^{10000}(roughly 10^{3000}); and confirmed, incidentally, that the mean number of iterations required for arbitrary odd n is of the order of $log_{\frac{4}{3}}(n)$; with the

[3] We wave our hands and assume the samples are independent. — More below.

[4] I think this is currently on the order of a quintillion.

worst-case exception of numbers of the form $2^m - 1$, which neatly double that estimate.[5]

Along one natural path of generalization the question rapidly devolves into a family of problems which can be shown to be unsolvable.[6] But there is another way to go about it, which seems rather more interesting.

(iv)

Let us introduce a few notational conveniences. First, we annotate the arrows with the number of divisions they involve:

$$x_0 \underset{\lambda_1}{\rightarrow} x_1 \underset{\lambda_2}{\rightarrow} \dots \underset{\lambda_n}{\rightarrow} x_n$$

so that, for instance

$$7 \underset{1}{\rightarrow} 11 \underset{1}{\rightarrow} 17 \underset{2}{\rightarrow} 13 \underset{3}{\rightarrow} 5 \underset{4}{\rightarrow} 1$$

Then we define

$$\sigma_0 = 0$$

$$\sigma_n = \sum_{k=0}^{n} \lambda_k$$

and note that it works out then that

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[5] A close examination of the statistics for numbers 2^n - 1 seems to show regularities which might make explicit formulae possible, but it is easy to hallucinate patterns in large tables of large numbers.

[6] See John Conway, "On Unsettleable Arithmetical Problems." *The American Mathematical Monthly*, **120**, No. 3 (March 2013), pp. 192-198. ("Unsettleable" is a neologism Conway introduces in this paper.)

$$x_0 = -\frac{1}{3}\sum_{k=0} \frac{2^{\sigma_k}}{3^k} = -\frac{1}{3}(1 + \frac{2^{\lambda_1}}{3}(1 + \frac{2^{\lambda_2}}{3}(1 + \ldots$$

in the sense of p-adic (here 2-adic) convergence; thus e.g. from

$$1 \underset{2}{\rightarrow} 1$$

it follows that

$$1 = -\frac{1}{3}\left(1 + \frac{4}{3}\left(1 + \frac{4}{3}(1 + \ldots)\right)\right) = -\frac{1}{3} \bullet \frac{1}{1 - \frac{4}{3}}$$

from

$$13 \underset{3}{\rightarrow} 5 \underset{3}{\rightarrow} 1$$

it follows that

$$13 = -\frac{1}{3}(1 + \frac{2^3}{3}\left(1 + \frac{2^4}{3}\left(\frac{1}{1 - \frac{4}{3}}\right)\right))$$

and so on.

Then it is not difficult to prove that:

[1] As the 2-adic series indicates, the closer two numbers are 2-adically,[7] the longer the agreement between their series of divisors;

[2] The series is eventually periodic if and only if the series of divisors is recurrent; thus

[3] The process cycles if and only if the series of divisors is recurrent.[8]

Moreover

[4] All of the $2^{\sigma-1}$ solutions in positive integers to the equation

$$\lambda_1 + \lambda_2 + \ldots + \lambda_k = \sigma$$

$$(1 \leq k \leq \sigma)$$

are represented by series of divisors for sequences

$$x_0 \underset{\lambda_1}{\rightarrow} x_1 \underset{\lambda_2}{\rightarrow} \ldots \underset{\lambda_n}{\rightarrow} x_n$$

beginning with exactly half of the odd numbers

$$x_0 < 2^{\sigma+1}$$

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[7] I.e. the larger the power of 2 that divides the difference between them. See Neal Koblitz, *P-adic Numbers, P-adic Analysis, and Zeta-functions*, Springer-Verlag, 1984.

[8] I had some intuitive inkling of this several years before I worked it out in detail, when one night after staggering home from a party I found I was too drunk to sleep and remembered this problem. It suddenly struck me very forcibly that the situation had to be analogous to that of the recurrent decimal expansion of a rational number, though I had no idea at the time how to derive any such expression. — It is one of the great mysteries about mathematical insight, that often you abruptly know the answer to something without having any idea what made you think of it or how you will prove it. (Of course we know how Plato explained this.)

The first few tables will illustrate the principle:

$$(\sigma = 1)$$

$$3 \underset{1}{\rightarrow} 5$$

$$(\sigma = 2)$$

$$1 \underset{2}{\rightarrow} 1$$

$$7 \underset{1}{\rightarrow} 11 \underset{1}{\rightarrow} 17$$

$$(\sigma = 3)$$

$$9 \underset{2}{\rightarrow} 7 \underset{1}{\rightarrow} 11$$

$$11 \underset{1}{\rightarrow} 17 \underset{2}{\rightarrow} 13$$

$$13 \underset{3}{\rightarrow} 5$$

$$15 \underset{1}{\rightarrow} 23 \underset{1}{\rightarrow} 35 \underset{1}{\rightarrow} 53$$

$$(\sigma = 4)$$

$$1 \underset{2}{\rightarrow} 1 \underset{2}{\rightarrow} 1$$

$$5 \underset{4}{\rightarrow} 1$$

$$7 \underset{1}{\rightarrow} 11 \underset{1}{\rightarrow} 17 \underset{2}{\rightarrow} 13$$

$$9 \underset{2}{\rightarrow} 7 \underset{1}{\rightarrow} 11 \underset{1}{\rightarrow} 17$$

$$19 \underset{1}{\rightarrow} 29 \underset{3}{\rightarrow} 11$$

$$27 \underset{1}{\rightarrow} 41 \underset{2}{\rightarrow} 31 \underset{1}{\rightarrow} 47$$

$$29 \underset{3}{\rightarrow} 11 \underset{1}{\rightarrow} 17$$

$$31 \underset{1}{\rightarrow} 47 \underset{1}{\rightarrow} 71 \underset{1}{\rightarrow} 107 \underset{1}{\rightarrow} 161$$

$(\sigma = 5)$

$$3 \underset{1}{\rightarrow} 5 \underset{4}{\rightarrow} 1$$

$$17 \underset{2}{\rightarrow} 13 \underset{3}{\rightarrow} 5$$

$$19 \underset{1}{\rightarrow} 29 \underset{3}{\rightarrow} 11 \underset{1}{\rightarrow} 17$$

$$27 \underset{1}{\rightarrow} 41 \underset{2}{\rightarrow} 31 \underset{1}{\rightarrow} 47 \underset{1}{\rightarrow} 71$$

$$33 \underset{2}{\rightarrow} 25 \underset{2}{\rightarrow} 19 \underset{1}{\rightarrow} 29$$

$$37 \underset{4}{\rightarrow} 7 \underset{1}{\rightarrow} 11$$

$$39 \xrightarrow[1]{} 59 \xrightarrow[1]{} 89 \xrightarrow[2]{} 67 \xrightarrow[1]{} 101$$

$$41 \xrightarrow[2]{} 31 \xrightarrow[1]{} 47 \xrightarrow[1]{} 71 \xrightarrow[1]{} 107$$

$$43 \xrightarrow[1]{} 65 \xrightarrow[2]{} 49 \xrightarrow[2]{} 37$$

$$45 \xrightarrow[3]{} 17 \xrightarrow[2]{} 13$$

$$47 \xrightarrow[1]{} 71 \xrightarrow[1]{} 107 \xrightarrow[1]{} 161 \xrightarrow[2]{} 121$$

$$53 \xrightarrow[5]{} 5$$

$$55 \xrightarrow[1]{} 83 \xrightarrow[1]{} 125 \xrightarrow[3]{} 47$$

$$57 \xrightarrow[2]{} 43 \xrightarrow[1]{} 65 \xrightarrow[2]{} 49$$

$$61 \xrightarrow[3]{} 23 \xrightarrow[1]{} 35 \xrightarrow[1]{} 53$$

$$63 \xrightarrow[1]{} 95 \xrightarrow[1]{} 143 \xrightarrow[1]{} 215 \xrightarrow[1]{} 323 \xrightarrow[1]{} 485$$

Similarly one can show that in a sequence

$$x_0 \xrightarrow[\lambda_1]{} x_1 \xrightarrow[\lambda_2]{} \dots \xrightarrow[\lambda_n]{} x_n$$

$$x_0 < x_n$$

if and only if

$$2^{\sigma} < 3^{n}$$

This creates the possibility of another handwaving probability argument: let ω be the smallest positive integer that does not reduce; then in any table in which ω appears, the above inequality must hold. But one may ask what fraction of the solutions

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = \sigma$$

have the property that

$$\sigma < n \bullet log_2(3)$$

when n is large, and a calculation shows that it approaches 0. In this sense we can assert that the likelihood of any particular integer being ω is vanishingly small.

(v)

The 2-adic series suggests a generalization to arbitrary p which takes the following form: observe that the handwaving probability argument works just as well (or badly) if we suppose a mapping

n —>(p+1)n + r

where r is the one of 1,...,(p-1) that guarantees divisibility by p. Then if we write a sequence of such mappings

$$x_n \to \frac{(p+1)x_n + r_n}{p^{l_{n+1}}} = x_{n+1}$$

as

$$x_0 \xrightarrow[r_0, l_1]{} x_1 \xrightarrow[r_1, l_2]{} x_2 \to \dots$$

then

$$x_0 = -\frac{1}{p+1} \sum_{n=0} r_n p^{s_n} (\frac{1}{p+1})^n$$

where again

$$s_0 = 0$$

$$s_n = \sum_{k=0}^{n} l_k$$

Whether p really needs to be prime to allow us to interpret the series as p-adically convergent is a question whose discussion may be either postponed or ignored; suffice it that for arbitrary p the fundamental loop is

1 —> 2 —> ... —> (p - 1) —> 1

and for p from 2 to 101 based on a preliminary survey this appears to be the unique target in the cases

p = 2, 5, 7, 8, 13, 14, 18, 19, 21, 22, 26, 28, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 55, 56, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 79, 80, 81, 82, 83, 84, 86, 87, 88, 89, 90, 91, 92, 93, 95, 96, 97, 98, 99, 100, 101, i.e. in 77 instances; and while though an exhaustive listing of the 23 others would probably constitute information overload, it is interesting to note that, in the four cases 12, 20, 24, and 54, there are at least two alternative terminal loops, and that the loop beginning on n = 3416 with p = 57 has 464 elements.

Thus we see that in general the natural conjecture is not that the target loop is unique, but that the number of such loops is finite.[9] Beyond that, however, it seems that a kind of caprice is at work, and it would be very difficult if not impossible to characterize exactly how the loops are determined.

The handwaving probability argument may be conservative, since the conjecture also appears to be true for some higher multipliers — e.g. the pairs (p + 2, p) for p less than 21, though of course with different targets; for (16,14) e.g. about ten percent of inputs arrow (1497 1711 1956 2236 2556 2922 3340 3818 4364 4988 5701 6516 7447 8511 9727 11117 12706 14522 16597 18969 21679 24777 28317 32363 36987 42271 48310 55212 63100 72115 5887 6729 7691 8790 10046 11482 13123 14998 17141 19590 22389 25588 29244 33422 38197 43654 49891 57019 65165 74475 85115 97275 111172 127054 145205 165949 189657 216751 17694 20222 23111 26413 30187 34500 39429 45062 51500 58858 67267 76877 87860 100412 114757 131151 149887 171300 195772 223740 255703 292233 333981 381693 436221 498539 569759 46511 53156 60750 69429 79348 90684 103639 118445 9669 11051 12630 14435 16498 18855 21549 24628 28147 32169 36765 42018 48021 54882 62723 71684 81925 93629 107005 122292 139763 159730 182549 14902 17031 19465 1589 1817 2077 2374 2714 3102 3546 4053 4633 5295 6052 6917 7906 9036 10327 11803 13490 15418 17621 20139 23017 1879 2148 2455 2806 3207 3666 4190 4789 391 447 511 585 669 765 875 1001 1145 1309). This suggests that some kind

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[9] Another natural question is how to distinguish those p for which the target loop is unique. Of course I have no idea how to do that.

of statistical law is at work, and that there is a kind of phase transition between multipliers that converge and multipliers that blow up whose position shifts with each divisor.[10]

(v.1)

If for the mapping

$$n \to \frac{(p+1)n + r}{p^l}$$

we define a function

$$f(n) = length(reduction(p+1, p, n))$$

then a reasonable approximation might be

$$f(n) \approx 1 + \frac{p-1}{p} f\left(\frac{p+1}{p} n\right) + \frac{p-1}{p^2} f\left(\frac{p+1}{p^2} n\right) + \ldots$$

a functional equation which is not obviously solvable at first glance. But consider the guess[11]

$$f(n) = \frac{1}{\lambda(p)} \log(n)$$

Then we have

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[10] One could try to locate this more precisely by further randomizing the mapping by selecting multipliers (p + 1), (p + 2), etc. according to a statistical distribution and determining the relationship between the mean multiplier and convergence behavior. But I have yet to be assailed by a fit of boredom of magnitude sufficient to motivate writing the code. Eventually, no doubt.

[11] Or "ansatz", if you want to pretend we aren't just bullshitting here.

$$\frac{1}{\lambda}\log(n) = 1 + \frac{p-1}{\lambda}(\frac{1}{p}\left(\log\left(\frac{p+1}{p}\right) + \log(n)\right)$$

$$+\frac{p-1}{\lambda}\frac{1}{p^2}\left(\log\left(\frac{p+1}{p^2}\right) + \log(n)\right) + \ldots$$

$$= 1 + \frac{p-1}{\lambda}(\frac{1}{p} + \frac{1}{p^2} + \ldots)(\log(n) + \log\left(p+1\right) - \frac{p}{\left(p-1\right)^2}\log(p)$$

since

$$\sum_{k=1}\frac{1}{p^k}\log(\frac{1}{p^k}) = -\log(p)\sum_{k=1}\frac{k}{p^k}$$

i.e.

$$\frac{1}{\lambda}\log(n) = 1 + \frac{1}{\lambda}\log(n) + \frac{1}{\lambda}\log\left(p+1\right) - \frac{1}{\lambda}\frac{p}{\left(p-1\right)}\log(p)$$

or

$$\lambda = \log(\frac{p^{\frac{p}{p-1}}}{p+1})$$

which for p = 2 does, indeed, imply that

$$\lambda = \log(4/3)$$

The empirical evidence for larger p is less conclusive, but appears to favor the hypothesis.

If for a given sequence

$$s = \{x_0, x_1, x_2, \dots\}$$

we define the derived sequence $\delta_n(s)$ by forming the differences

$$\begin{aligned}
\delta_0 &= x_0 \\
\delta_1 &= x_0 - x_1 \\
\delta_2 &= x_0 - 2x_1 + x_2 \\
&\dots \\
\delta_n &= \sum_{k=0}^{n} (-1)^k C_k^n x_k
\end{aligned}$$

where C_k^n are as usual the binomial coefficients, then the Euler transformation may be defined by the equivalence

$$\sum_{n=0} (-1)^n x_n z^n = \frac{1}{1+z} \bullet \sum_{n=0} \delta_n(x) (\frac{z}{1+z})^n$$

Applying this to the series

$$\begin{aligned}
x &= -\frac{1}{3} \sum_{n=0} 2^{\sigma_n} (\frac{1}{3})^n \\
&= -\frac{1}{3} \sum_{n=0} (-1)^n 2^{\sigma_n - n} (\frac{-2}{3})^n
\end{aligned}$$

we obtain

$$x = -\sum_{n=0} \delta_n(2^{\sigma_n - n})(-2)^n$$

Moreover we know from p-adic analysis that

$$2^{\sigma_n}$$

is recurrent if and only if

$$\sum_{n=0} 2^{\sigma_n}$$

is rational; again applying the Euler transform this is

$$-\sum_{n=0} \delta_n(2^{\sigma_n - n})2^n$$

From this we see that the proposition we are trying to establish is that, for any n, the derived function

$$f_n(z) = \sum_{k=0} \delta_k(2^{\sigma_k - k})z^k$$

has the property that $f_n(2)$ is rational if and only if $f_n(-2)$ is rational; more, if $f_n(2)$ is a rational number, then $f_n(z)$ is a rational function.

Which neglects many details, but — well, more anon.

(vii)

As for the girl, we talked a while longer and then lapsed into silence. We were shy, I suppose, not looking at one another directly. But since we were sitting at the bar, I could look up and glance into the mirror behind it to examine her. And when I did this I could see she was looking at me. — The people in the mirror had something rather different to say to one another, I thought. — Perhaps I anticipated what Otto Rank would tell me about *The Student of Prague*. Or perhaps I remembered what Borges claimed Bioy Casares had told him about the heresiarchs of Tlön: that mirrors and copulation were abominable, because they multiplied the numbers of mankind.

Whatever. We ended up living together for seven years.